Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(02)00004-7
J. Appl. Maths Mechs, Vol. 66, No. 1, pp. 23–30, 2002

0021-8928/02/\$—see front matter

THE BOUNDEDNESS OF TRAJECTORIES IN THE NEIGHBOURHOOD OF THE ORBITALLY UNSTABLE PERIODIC MOTION OF A HAMILTONIAN SYSTEM†

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(*Received* 17 *April* 2001)

An autonomous Hamiltonian system with two degrees of freedom is considered. It is assumed that a periodic motion and a second-order resonance (parametric resonance) exist in the system. The unperturbed periodic motion is orbitally stable or unstable. However, even in the case of instability, the trajectories of the perturbed motion may remain in a bounded neighbourhood of the unperturbed trajectory for all values of the time. An asymptotic estimate of the size of this neighbourhood is given for the case when the Hamiltonian depends on a small parameter. The results are applied to the problem of the non-local stability of fast planar rotations of a heavy rigid body in the Kovalevskaya case, and to the problem of the stability of periodic Poincaré motions of the first kind in the restricted three-body problem, for one special case of second-order resonance.

1. STATEMENT OF THE PROBLEM

Consider an autonomous Hamiltonian system with two degrees of freedom. Suppose it has a periodic motion and the Hamiltonian Γ is analytic in the neighbourhood of a trajectory corresponding to that motion. We may assume without loss of generality that the period is 2π.

Canonically conjugate variables ξ_i and η_i (ξ_i are the coordinates and η_i are the momenta; $i = 1, 2$) can be chosen [1] in such a way that the solution corresponding to unperturbed motion may be written in the form

$$\xi_1(t) = t + \xi_1(0), \quad \eta_1 = \xi_2 = \eta_2 = 0 \tag{1.1}$$

and the Hamiltonian will be a 2π-periodic function of the coordinate ξ_1.

The problem of the orbital stability of the unperturbed periodic motion is equivalent to the problem of stability with respect to perturbations of the variables η_1, ξ_2, η_2.

Suppose the Hamiltonian depends on a parameter ε and is analytic for sufficiently small values of that parameter. We shall also assume that when $\varepsilon = 0$ the function Γ is independent of ξ_1.

Two characteristic exponents of the linearized equations of perturbed motion will always vanish. As regards the two others, we shall assume that when $\varepsilon = 0$ they are pure imaginary numbers $\pm i\omega$, where 2ω is close to an odd integer $2n + 1$. Let us put

$$2n + 1 - 2\omega = 2\varepsilon\alpha \tag{1.2}$$

In the case of this second-order (parametric) resonance, the unperturbed periodic motion may be orbitally stable or unstable, with instability possibly observed already in the linearized equations of perturbed motion. However, this instability may turn out to be only local, since the trajectories of the perturbed motion do not necessarily depart to an unbounded distance from the trajectory of the unperturbed motion. They may remain perpetually in some bounded (though not infinitesimal) neighbourhood of that trajectory. The main aim of this paper is to derive estimates for the size of that neighbourhood.

†*Prikl. Mat. Mekh.* Vol. 66, No. 1, pp. 24–32, 2002.

2. THE NORMAL FORM OF THE HAMILTONIAN OF PERTURBED MOTION AND ITS SIMPLIFICATION

Let us express the Hamiltonian Γ of perturbed motion as a series in powers of η_1, ξ_2, η_2, ε. Then a suitable canonical change of variables $\xi_1, \eta_1, \xi_2, \eta_2, \to \varphi_1, r_1, q_2, p_2$ 2π-periodic in φ_1 and analytic in $\eta_1, \xi_2, \eta_2, \varepsilon$, may be used to annihilate all non-resonance terms in the part of the Hamiltonian that is quadratic in q_2 and p_2, completely annihilating terms of the third and also (when $\varepsilon = 0$) the fifth degree in $|r_1|^{1/2}, q_2, p_2$, and reducing terms of the fourth degree to a form such that, when $\varepsilon = 0$, they depend only on r_1 and on the combination $q_2^2 + p_2^2$. The Hamiltonian of the perturbed motion, normalized in this way, will be denoted by H. It has the following form (see [2]).

$$H = r_1 + \tfrac{1}{2}\lambda(q_2^2 + p_2^2) + \tfrac{1}{2}\varepsilon[\varkappa_1 \sin 2\varphi - \varkappa_2 \cos 2\varphi](q_2^2 - p_2^2) + \varepsilon[\varkappa_1 \cos 2\varphi +$$

$$+ \varkappa_2 \sin 2\varphi] q_2 p_2 + c_{20} r_1^2 + \tfrac{1}{2} c_{11} r_1 (q_2^2 + p_2^2) + \tfrac{1}{4} c_{02}(q_2^2 + p_2^2)^2 + \sum_{k=1}^{\infty} \varepsilon^k H_4^{(k)} +$$

$$+ \sum_{k=1}^{\infty} \varepsilon^k H_5^{(k)} + O_6 \tag{2.1}$$

where $\varphi = \tfrac{1}{2}(2n+1)\varphi_1$, $H_4^{(k)}$ and $H_5^{(k)}$ are forms of degree four and five in $|r_1|^{1/2}, q_2, p_2$ whose coefficients are 2π-periodic in φ_1, and O_6 denotes the totality of terms of degree at least six in $|r_1|^{1/2}, q_2, p_2$. The quantities $\lambda, \varkappa_1, \varkappa_2, c_{20}, c_{11}, c_{02}$ are constant coefficients, of which $\lambda, \varkappa_1, \varkappa_2$ are represented by convergent series

$$\lambda = \omega + \varepsilon\lambda^{(1)} + \varepsilon^2\lambda^{(2)} + \ldots, \quad \varkappa_i = \varkappa_i^{(1)} + \varepsilon\varkappa_i^{(2)} + \ldots, \quad i = 1, 2$$

Formulae defining the quantities c_{ij} and $\varkappa_i^{(1)}$ in terms of the expansion coefficients of the initial Hamiltonian Γ were obtained in [2].

If we now apply the canonical transformation

$$\varphi_1 = \tilde{\varphi}_1, \quad r_1 = \tilde{r}_1 - \tfrac{1}{4}(2n+1)(\tilde{q}_2^2 + \tilde{p}_2^2) \tag{2.2}$$

$$q_2 = \tilde{q}_2 \cos\varphi + \tilde{p}_2 \sin\varphi, \quad p_2 = -\tilde{q}_2 \sin\varphi + \tilde{p}_2 \cos\varphi$$

we obtain a Hamiltonian

$$H = \tilde{r}_1 - \tfrac{1}{2}\varepsilon\delta(\tilde{q}_2^2 + \tilde{p}_2^2) + \tfrac{1}{2}\varepsilon[2\varkappa_1\tilde{q}_2\tilde{p}_2 + \varkappa_2(\tilde{p}_2^2 - \tilde{q}_2^2)] + a_{20}\tilde{r}_1^2 + \tfrac{1}{2}a_{11}\tilde{r}_1(\tilde{q}_2^2 + \tilde{p}_2^2) +$$

$$+ \tfrac{1}{4}a_{02}(\tilde{q}_2^2 + \tilde{p}_2^2)^2 + \sum_{k=1}^{\infty} \varepsilon^k \tilde{H}_4^{(k)} + \sum_{k=1}^{\infty} \varepsilon^k \tilde{H}_5^{(k)} + O_6$$

where $\tilde{H}_4^{(k)}$ and $\tilde{H}_5^{(k)}$ are the forms $H_4^{(k)}$ and $H_5^{(k)}$ of (2.1), expressed in terms of the new variables, and

$$\delta = \alpha - \lambda^{(1)} - \varepsilon\lambda^{(2)} - \ldots, \quad a_{20} = c_{20}$$
$$a_{11} = c_{11} - (2n+1)c_{20}, \quad a_{02} = c_{20}(n+1/2)^2 - c_{11}(n+1/2) + c_{02}$$

One more canonical transformation (with valence $a_{02}(\varepsilon\varkappa)^{-1}$)

$$\tilde{\varphi}_1 = \sigma w_1, \quad \tilde{r}_1 = \varepsilon\varkappa |a_{02}|^{-1} I_1 \tag{2.3}$$

$$\tilde{q}_2 = (2\varepsilon\varkappa |a_{02}|^{-1}\rho_2)^{1/2} \sin(\psi + \psi_0), \quad \tilde{p}_2 = (2\varepsilon\varkappa |a_{02}|^{-1}\rho_2)^{1/2} \cos(\psi + \psi_0)$$

where

$$\varkappa = (\varkappa_1^{(1)^2} + \varkappa_2^{(1)^2})^{1/2}, \quad \varkappa_1^{(1)} = \varkappa \sin 2\psi_0, \quad \varkappa_2^{(1)} = \varkappa \cos 2\psi_0, \quad \psi = \sigma\theta_2 + \tfrac{1}{4}(1-\sigma)\pi,$$

$$\sigma = \operatorname{sign} a_{02}$$

and a change to a new independent variable $\rho = \varkappa t$ lead to the equations of perturbed motion with Hamiltonian

$$H = H^{(0)}(I_1) + \varepsilon H^{(1)}(I_1, \rho_2, \theta_2) + \varepsilon^2 H^{(2)}(I_1, \rho_2, w_1, \theta_2; \varepsilon^{1/2}) \tag{2.4}$$

where

$$H^{(0)} = \sigma \varkappa^{-1} I_1, \quad H^{(1)} = -\nu\rho_2 + \rho_2 \cos 2\theta + \rho_2^2 + b_{11} I_1 \rho_2 + b_{20} I_1^2 \tag{2.5}$$

$$\nu = \sigma(\alpha - \lambda^{(1)})\varkappa^{-1}, \quad b_{11} = a_{11} a_{02}^{-1}, \quad b_{20} = a_{20} a_{02}^{-1}$$

Note that the quantity I_1 may have any sign, but $\rho_2 \geqslant 0$. The function $H^{(2)}$ is 4π-periodic in w_1, 2π-periodic in θ_2 and is analytic in all its arguments for $\rho_2 > 0$.

If $-1 < \nu \leqslant 1$, the periodic motion (1.1) is orbitally unstable; if $\nu \leqslant -1$ or $\nu > 1$, it is stable [2].

3. THE APPROXIMATE SYSTEM

If the last term is dropped in Hamiltonian (2.4), we arrive at an approximate system with Hamiltonian $H^{(0)} + \varepsilon H^{(1)}$. In this system I_1 is constant ($I_1 = I_1(0)$) and the variables θ_2 and ρ_2 are described by the equations

$$\frac{d\theta_2}{d\tau} = \varepsilon \frac{\partial \gamma}{\partial \rho_2}, \quad \frac{d\rho_2}{d\tau} = -\varepsilon \frac{\partial \gamma}{\partial \theta_2} \tag{3.1}$$

where

$$\gamma = -\tilde{\nu}\rho_2 + \rho_2 \cos 2\theta_2 + \rho_2^2, \quad \tilde{\nu} = \nu - b_{11} I_1 \tag{3.2}$$

The quantity ν in Hamiltonian (2.4) may be treated as a parameter characterizing the "resonance mismatch." The quantity $\tilde{\nu}$ in the function (3.2) also plays the role of a resonance mismatch. However, unlike ν, which depends only on the parameters of the system being investigated, $\tilde{\nu}$ also depends on the initial data (on $I_1(0)$). If $b_{11} \neq 0$, the resonance mismatch may be increased, decreased or even completely annihilated by a suitable choice of $I_1(0)$.

The system of equations (3.1) has been studied in detail previously (see [3, 4] and the references listed there). Its trajectories may be qualitatively different for different values of the parameter $\tilde{\nu}$, that is, the presence of a degree of freedom corresponding to the variables w_1 and I_1 exerts a considerable influence on the behaviour of the solutions $\theta_2(\tau)$ and $\rho_2(\tau)$ of system (3.1).

Phase portraits of system (3.1) in the x_1, x_2 plane, where $x_1 = (2\rho_2)^{1/2} \cos \theta_2$, $x_2 = (2\rho_2)^{1/2} \sin \theta_2$, are shown in Fig. 1(a–c), for the cases $\tilde{\nu} \leqslant -1$, $-1 < \tilde{\nu} \leqslant 1$, $\tilde{\nu} > 1$, respectively. Depending on the value of

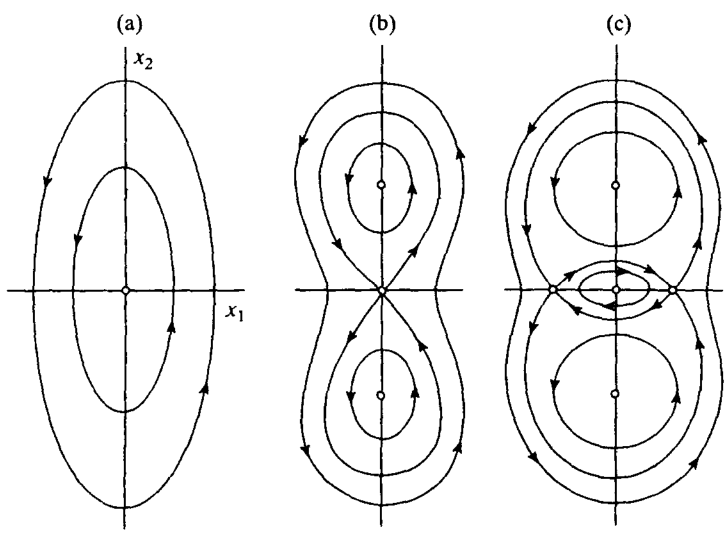

Fig. 1

$\tilde{\nu}$, the system may have one, three or five singular points. The singular point for which $x_1 = x_2 = 0$ exists for any value of $\tilde{\nu}$. It corresponds to the unperturbed periodic motion (1.1).

In what follows we will be interested only in domains of the x_1, x_2 plane filled by trajectories encircling all the singular points of system (3.1). For any value of $\tilde{\nu}$, these trajectories correspond to positive values of the constant h of the integral

$$\gamma(\rho_2, \theta_2) = h \tag{3.3}$$

Previous analysis [3, 4] implies that, if the following condition holds at $\tau = 0$

$$\rho_2(0) < \tfrac{1}{2}[\tilde{\nu} - 1 + \sqrt{(\tilde{\nu} - 1)^2 + a}] \tag{3.4}$$

then for all $\tau \geq 0$ we have the inequality

$$\rho_2(\tau) < \tfrac{1}{2}[\tilde{\nu} + 1 + \sqrt{(\tilde{\nu} + 1)^2 + a}] \tag{3.5}$$

The quantity a in inequalities (3.4) and (3.5) is a positive parameter ($a = 4h$, where h is the constant of the integral (3.3) corresponding to a closed trajectory encircling all the singular points of system (3.1); the right-hand sides of (3.4) and (3.5) are the minimum and maximum values of ρ_2 on that trajectory).

4. ACTION-ANGLE VARIABLES

Let us return now to the exact equations of perturbed motion as defined by Hamiltonian (2.4), writing it in terms of the new variables I_i and w_i ($i = 1, 2$), which are action-angle variables for the approximate system. Since w_1 is a cyclic coordinate in the approximate system, one pair of such variables is I_1 and w_1. Denote Hamiltonian (2.4), written in terms of the variables I_i and w_i, by F

$$F = F^{(0)}(I_1) + \varepsilon F^{(1)}(I_1, I_2) + \varepsilon^2 F^{(2)}(I_1, I_2, w_1, w_2; \varepsilon^{1/2}) \tag{4.1}$$

where $F^{(0)}$ is the function $H^{(0)}$ of (2.5)

$$F^{(1)} = b_{20} I_1^2 + \Phi(I_1, I_2)$$

and Φ is the function (3.2) written in terms of the new variables; it is the inverse of the function

$$I_2(h) = \frac{1}{2\pi} \oint \rho_2(\theta_2, h) d\theta \tag{4.2}$$

where ρ_2 is the value of the momentum ρ_2 on trajectories of the approximate system for $h > 0$; the dependence of ρ_2 on I_1 is not shown in (4.2).

The function F is 4π-periodic in w_1 and 2π-periodic in w_2; for $I_2 > 0$ it is analytic in I_1, I_2, w_1, w_2, $\varepsilon^{1/2}$.

5. THE VARIABLES I_1 AND I_2 HAVE NO EVOLUTION

When $\varepsilon = 0$ the Hamiltonian (4.1) depends only on one of the "action" variables (namely, I_1), so that we have proper degeneracy [5]. When $0 < \varepsilon \ll 1$, however, the degeneracy is removed, and the variables I_1 and I_2 in the perturbed system always remain near their initial values. Indeed, one can verify (see below) that Hamiltonian (4.1) satisfies the conditions

$$\frac{\partial F^{(0)}}{\partial I_1} \neq 0, \quad \frac{\partial F^{(1)}}{\partial I_2} \neq 0, \quad \frac{\partial^2 F^{(1)}}{\partial I_2^2} \neq 0 \tag{5.1}$$

and therefore [5], for most initial data, the motion in the perturbed system will be conditionally periodic. Only a fraction $O(\exp(-c_1 \varepsilon^{-1}))$ where $c_1 = \text{const} > 0$, of the phase space is not filled by conditionally periodic trajectories. Moreover, for all initial data the quantities $I_i(\tau)$ ($i = 1, 2$) are close to their initial values

$$|I_i(\tau) - I_i(0)| < c_2 \varepsilon \quad (c_2 = \text{const}) \tag{5.2}$$

The truth of the first condition of (5.1) follows at once from (2.5) and (4.1), since $\partial F^{(0)}/\partial I_1 = \sigma\varkappa^{-1} \neq 0$. The second inequality in (5.1) is also true, because $\partial F^{(1)}/\partial I_2$ is the frequency, divided by ε, of motion along the closed trajectory considered in Fig. 1, which is, of course non-zero. As regards $\partial^2 F^{(1)}/\partial I_2^2$ in the third condition of (5.1), it follows from (3.3) and (4.2) that

$$\frac{\partial^2 F^{(1)}}{\partial I_2^2} = \frac{1}{2\pi}\left(\frac{\partial F^{(1)}}{\partial I_2}\right)^3 \oint \left(\frac{\partial \gamma}{\partial \rho_2}\right)^{-3} \frac{\partial^2 \gamma}{\partial \rho_2^2} d\theta_2 \tag{5.3}$$

The integrand in (5.3) is positive on the closed trajectories under consideration, which encircle all the singular points of system (3.1) in Fig. 1. Hence it follows that the third condition of (5.1) is also true. This proves the assertion that the quantities I_1 and I_2 have no evolution.

6. ESTIMATES OF $r_1(t)$ AND $q_2^2(t) + p_2^2(t)$ IN THE CASE WHEN $b_{11} = 0$

It follows from inequality (5.2) and formulae (2.2) and (2.3) for the replacement of variables that, for all $t > 0$

$$r_1 + \tfrac{1}{4}(2n+1)(q_2^2 + p_2^2) = \varepsilon\varkappa\,|a_{02}|^{-1} I_1 = \varepsilon\varkappa\,|a_{02}|^{-1} (I_1(0) + O(\varepsilon)) \tag{6.1}$$

Hence we may conclude that $r_1 + \tfrac{1}{4}(2n + 1)(q_2^2 + p_2^2)$ is "almost an integral" of the equations of perturbed motion, in the sense that for all $t > 0$ it differs from its initial value by a quantity of order of magnitude at least two in ε, provided that $r_1(0)$ and $q_2^2(0) + p_2^2(0)$ are of order of magnitude at least one.

With relation (6.1) available, the derivation of an estimate for the size of the neighbourhood in which there are trajectories of perturbed motion reduces to finding estimates for the quantity $q_2^2(t) + p_2^2(t)$.

We will first consider the case when the coefficient b_{11} in Hamiltonian (2.4) vanishes. In that case, by the second equality of (3.2), we have $\bar{\nu} = \nu$. Then, taking into account that $I_2(t) = I_2(0) + O(\varepsilon)$ for all initial data, (see inequality (5.2)), we deduce from formulae (2.3) and inequalities (3.4) and (3.5) that if

$$q_2^2(0) + p_2^2(0) < \varepsilon\varkappa\,|a_{02}|^{-1} [\nu - 1 + \sqrt{(\nu - 1)^2 + a} - \delta_1] \tag{6.2}$$

then for all $t \geqslant 0$

$$q_2^2(t) + p_2^2(t) < \varepsilon\varkappa\,|a_{02}|^{-1} [\nu + 1 + \sqrt{(\nu + 1)^2 + a} + \delta_2] \tag{6.3}$$

where δ_i are positive numbers of the same order of magnitude as $\chi = \varepsilon^{1-\beta}$ $(0 < \beta < 1)$.

Inequalities (6.2) and (6.3) yield a one-parameter family of estimates. The parameter is the number a.

7. THE NON-LOCAL STABILITY OF FAST PLANAR ROTATIONS OF A RIGID BODY IN THE KOVALEVSKAYA CASE

Consider a rigid body of weight mg rotating about a fixed point O. The axes of a coordinate system $Oxyz$ fixed in the body are directed along the principal axes of inertia about O. The corresponding principal moments of inertia are denoted by A, B and C. Let the geometry of the masses of the body correspond to the Kovalevskaya case. Then, setting $B = C = 2A$, we may assume that the centre of gravity is on the Oz axis. The distance from the centre of gravity to the fixed point O is denoted by l.

Let us assume that the projection of the angular momentum of the body onto the vertical is zero. The body may perform pendulum-like rotations about the Ox axis, which is in a fixed horizontal position. Suppose the mean angular velocity Ω of this rotation is so large that the dimensionless quantity $\varepsilon = mgl/(A\Omega^2)$ may be considered to be a small parameter. It has been shown [6, 7] that in the Kovalevskaya case such planar rotations of a rigid body are orbitally unstable. An estimate for the size of the neighbourhood of the planar rotation in which the trajectories of the perturbed motion always remain may be obtained, by using the fact that the equations of motion are completely integrable (using the Kovalevskaya integral). Here, to illustrate the algorithm of Section 7, we will derive this estimate for sufficiently small ε.

As canonical conjugate variables we take Andoyer variables [8]. The generalized momenta, transformed to dimensionless form using the factor $A\Omega$, are denoted by R_1 and R_2 (R_1 is the absolute value of the angular momentum and R_2 its projection onto the Oz) axis; the corresponding generalized coordinates are denoted by ψ_1 and ψ_2, and the independent variable is taken to be $\tau = \Omega t$. The Hamiltonian has the form

$$H = \tfrac{1}{4}(R_1^2 - R_2^2)(1+\sin^2\psi_2) + \tfrac{1}{4}R_2^2 - \varepsilon\sqrt{1 - R_2^2/R_1^2}\cos\psi_1 \tag{7.1}$$

In the aforementioned rotations of the body, $\psi_2 = \pi/2$, $R_2 = 0$, and the variation of the variables ψ_1 and R_1 is described by equations with Hamiltonian $h_0 = \tfrac{1}{2}R_1^2 - \varepsilon\cos\psi$. In terms of the variables x_1 and X_1 introduced [4] by the canonical transformation

$$\psi_1 = x_1 + \varepsilon X_1^{-2}\sin x_1 + O(\varepsilon^2), \quad R_1 = X_1 + \varepsilon X_1^{-1}\cos x_1 + O(\varepsilon^2)$$

we have $h_0 = \tfrac{1}{2}X_1^2 + O(\varepsilon^2)$, and the planar rotation of the body is given by the equations

$$x_1(\tau) = (1 + O(\varepsilon^2))\tau + x_1(0), \quad X_1 = 1, \quad \psi_2 = \pi/2 \quad R_2 = 0$$

We introduce perturbations of Y_1, x_2, Y_2 by the formulae

$$X_1 = 1 - Y_1, \quad \psi_2 = \pi/2 + x_2, \quad R_2 = -Y_2$$

The Hamiltonian of the perturbed motion may be expressed as a series

$$H = Y_1 + \frac{1}{4}(x_2^2 + Y_2^2) + \frac{1}{2}\varepsilon\cos x_1(x_2^2 - Y_2^2) - \frac{1}{12}(x_2^4 + 3x_2^2Y_2^2 + 6Y_1x_2^2 + 6Y_1^2) + \ldots \tag{7.2}$$

where the dots stand for the totality of terms of degree more than five in $|Y_1|^{1/2}, x_2, Y_2$. Terms of order not less than the second and first in ε, which will not be needed in what follows, are omitted from the terms of the second and fourth degree, respectively, in expansion (7.2). It is obvious from (7.2) that there is a second-order resonance; in the notation of Section 1, we have $\omega = 1/2$, $n = 0$, $\alpha = 0$.

A canonical change of variables via the formulae

$$x_1 = x_1', \quad Y_1 = Y_1' - \tfrac{1}{4}\varepsilon\cos x_1'(x_2'^2 - Y_2'^2)$$

$$x_2 = x_2' - \tfrac{1}{2}\varepsilon\sin x_1'Y_2', \quad Y_2 = Y_2' - \tfrac{1}{2}\varepsilon\sin x_1'x_2'$$

reduces the first three terms in (7.2) (which correspond to the linearized equations of perturbed motion) to normal form. One further non-linear change of variables $x_1', x_2', Y_1', Y_2' \to \varphi_1, q_2, r_1, p_2$, close to the identical transformation, of the same type of Birkhoff's transformation, yields the normal form of the term of fourth degree in $|Y_1'|^{1/2}, x_2', Y_2'$. As a result we arrive at Hamiltonian (2.1) with

$$\varkappa_1^{(1)} = 0, \quad \varkappa_2^{(1)} = -\tfrac{1}{2}, \quad \lambda^{(1)} = 0, \quad c_{20} = c_{11} = 2c_{02} = -\tfrac{1}{2}.$$

By the formulae of Section 2, we find that $\varkappa = \tfrac{1}{2}$, $\nu = 0$, $b_{11} = 0$. Also, by relations (6.2) and (6.3), we see that if for $t = 0$

$$q_2^2(0) + p_2^2(0) < 4\varepsilon(\sqrt{1+a} - 1 - \delta_1)$$

then, for all t, we have the estimate

$$q_2^2(t) + p_2^2(t) < 4\varepsilon(\sqrt{1+a} + 1 + \delta_2)$$

It now follows from (6.1) that

$$|r_1(t) - r_1(0)| < 2\varepsilon(\sqrt{1+a} + O(\chi))$$

8. ESTIMATES IN THE CASE WHEN $b_{11} \neq 0$

The estimates given by inequalities (6.2) and (6.3) remain valid when $b_{11} \neq 0$, except that ν in these inequalities must be replaced by the quantity $\tilde{\nu}$ defined by the second equality of (3.2).

Taking into consideration that, by inequality (5.2), $I_1 = I_1(0) + O(\varepsilon)$, we may write the second equality of (3.2) in the form $\tilde{\nu} = \nu - b_{11}I_1(0) + O(\varepsilon)$. Using the transformation formulae (2.2) and (2.3) and introducing the notation

$$r_1(t) = \varepsilon\varkappa\,|\,a_{02}\,|^{-1}\,x(t),\quad q_2^2(t)+p_2^2(t) = \varepsilon\varkappa\,|\,a_{02}\,|^{-1}\,y(t),\quad x(0)=x_0,\quad y(0)=y_0 \tag{8.1}$$

we see that, with an error of order ε,

$$\tilde{\nu} = \tilde{\nu}(x_0, y_0) = \nu - b_{11}[x_0 + \tfrac{1}{4}(2n+1)y_0] \tag{8.2}$$

The quantity x may have any sign, but $y \geqslant 0$.

Suppose the parameters of the material system under investigation are given. Consequently, the quantities n, ν and b_{11} on the right-hand side of (8.2) are also given. Now fix some positive value of the parameter a and consider the inequality

$$y_0 < \tilde{\nu} - 1 + \sqrt{(\tilde{\nu}-1)^2 + a} \tag{8.3}$$

where $\tilde{\nu}$ is given by formula (8.2). In the half-plane $y_0 \geqslant 0$, inequality (8.3) will hold in some domain G. Suppose the initial date are such that $x_0 = x_0^*$, $y_0 = y_0^*$, and the point (x_0^*, y_0^*) lies in G together with its χ-neighbourhood. It then follows from inequalities (6.2) that for these initial data, for all $t \geqslant 0$, we have

$$y(t) < \nu^* + 1 + \sqrt{(\nu^*+1)^2 + a} + \delta_2 \tag{8.4}$$

on the trajectory of perturbed motion, where ν^* is the value of the function $\tilde{\nu}$ in (8.2) evaluated at the point (x_0^*, y_0^*).

9. THE QUESTION OF THE BOUNDEDNESS OF THE ORBITS OF ASTEROIDS IN THE HESTIA GAP

The distribution of asteroids according to their mean motions within the main asteroid belt, which lies between the orbits of Mars and Jupiter, is not uniform: there are intervals of distribution in which there are few asteroids or even none at all [9]. These intervals are known as gaps. For these gaps in the asteroid belt, the ratio of the mean motion of the asteroid to the mean motion of Jupiter is approximately a rational number $l : s$. For asteroids in the Hestia gap this ratio is 3 : 1. The number $l - s$ is known as the order of resonance, so that in the case of the asteroids of the Hestia gap one has the kind of resonance studied in this article – second-order resonance.

We shall simulate asteroid orbits by means of periodic Poincaré solutions of the first kind of the restricted circular three-body problem. Following Section 8, we shall estimate the size of the domain in which perturbed orbits are bounded near the unperturbed Poincaré orbits.

The units of measurement will be chosen so that the period of Jupiter's revolution, the distance between the Sun and Jupiter, and the sum of their masses are equal to unity. Jupiter's mass is denoted by ε. When ε = 0 the Poincaré orbits become circular orbits of radius R_0. For asteroids in the Hestia gap $R_0 = 9^{-1/3} = 0.48075$. The family of periodic Poincaré solutions depends (see, for example [10]) on one essential parameter h_1 (the quantity εh_1 is equal, with an error of order ε^2, to the difference between the constants of the energy integral in the Poincaré orbit and in the generating circular orbit). The normal form (2.1) of the Hamiltonian of perturbed motion in the neighbourhood of periodic Poincaré motions at second-order resonance was obtained in [10], where the meaning of the quantities r_1, q_2 and p_2 was also described. One can say that the perturbations q_2 and p_2 characterize the magnitude of the radial displacement and radial velocity, while r_1 is the magnitude of the perturbation of the transversal velocity of the asteroid in a synodic coordinate system.

Computations using the formulae developed in [10] for the coefficients of the normal form show that

$$\lambda = \tfrac{3}{2} + \varepsilon(-0.23747 + 1.62253h_1) + O(\varepsilon^2)$$

$$\varkappa_1^{(1)} = 0,\quad \varkappa_2^{(1)} = 0.86356,\quad c_{20} = \tfrac{1}{2}c_{11} = c_{02} = -3.24506$$

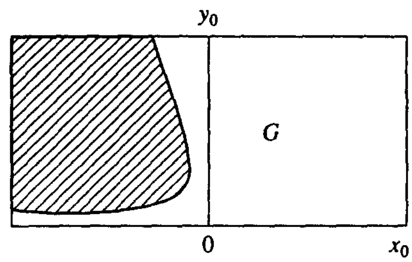

Fig. 2

The formulae of Section 2 now yield

$$a_{20} = -a_{11} = 4a_{02} = -3.24506, \quad \varkappa = 0.86356, \quad \sigma = -1, \quad b_{11} = -4$$

and expression (8.2) becomes

$$\tilde{\nu} = \tilde{\nu}(x_0, y_0) = \nu + 4x_0 + 3y_0 \tag{9.1}$$

The domain G is defined by relations (8.3) and (9.1). It is illustrated schematically in Fig. 2, with the part of the half-plane $y_0 = 0$ not belonging to G shown hatched. The boundary of G is the straight line $y_0 = 0$ and the hyperbola

$$x_0 = -(5y_0^2 + a)/(8y_0) - \tfrac{1}{4}(\nu - 1)$$

Suppose the initial values of r_1, q_2 and p_2 are such that the point (x_0^*, y_0^*), where

$$x_0^* = 0.939445\varepsilon^{-1} r_1(0), \quad y_0^* = 0.939445\varepsilon^{-1}(q_2^2(0) + p_2^2(0)) \tag{9.2}$$

lies inside the domain G and its distance from the latter's boundary is not less than χ. Then for all $t \geqslant 0$,

$$q_2^2(t) + p_2^2(t) < 1.06447\varepsilon[\nu^* + 1 + \sqrt{(\nu^* + 1)^2 + a} + \delta_2]$$

where ν^* is the value of the function (9.1) at the point (x_0^*, y_0^*) with coordinates (9.2).

The estimate for $r_1(t)$ may now be derived from (6.1).

This research was supported financially by the Russian Ministry of Education (E 00-11.0.28).

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Translated by D.L.